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On Benjamin's theory of conjugate vortex flows

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Benjamin (1962) introduced the idea that a given 'primary' swirling flow, with cylindrical stream surfaces, may have associated with it 'conjugate flows', also swirling and cylindrical, which in a certain sense are equivalent to the primary one. He deduced that, in cases where such conjugate flows exist and where the primary flow cannot support standing waves of small amplitude, the conjugate flow nearest the primary one (*a*) can support such waves, and (*b*) has a 'flow force' greater than that of the primary flow. In the present paper these two results are proved rigorously by a method which differs from Benjamin's.

1. Introduction

This paper is concerned with certain assertions made by Benjamin (1962) in his bold and important work on the phenomenon of vortex breakdown. Benjamin obtained his results by a mixture of physical reasoning, plausible argument, and appeal to advanced theorems in the calculus of variations. However, certain mathematical aspects of this work are open to criticism.† It would not be profitable to dwell on such criticism, because the arguments in Benjamin's paper are plausible enough and the results in question are correct in all essentials, but it does seem worth while to derive these results by a different method, which makes no appeal to the calculus of variations. (The set of functions admitted into competition by the variational approach is in fact much larger than is necessary for the task at hand.) We begin by sketching the essential ideas of the theory.

Let (r, ϑ, z) be cylindrical co-ordinates and (u, v, w) the corresponding components of fluid velocity. We write $\frac{1}{2}r^2 = y$, and consider the steady axisymmetric swirling flow of an inviscid fluid, of uniform density ρ, in a pipe of radius $(2a)^{\frac{1}{2}}$ and of infinite length in the z-direction. The stream function Ψ is defined by $\Psi_y = w$, $\Psi_z = -ru$, and by $\Psi = 0$ on $y = 0$.

Given that far upstream there is a 'primary' cylindrical flow A, with velocity

† For example, Benjamin calculated the difference in flow force of two conjugate flows by an application of Weierstrass's theorem (Bolza 1961) to what in our notation becomes the curve $\lambda = \lambda_{-1}$ in the (y, ψ)-plane of figure 2 (of the present paper). Now Weierstrass's theorem is applicable only to curves in a certain neighbourhood of the curve $\lambda = \lambda_0$; this neighbourhood may or may not extend to the curve $\lambda = \lambda_{-1}$. It does so for the extremals drawn in Benjamin's paper and in figure 2; but, if in the (y, λ)-plane of figure 2 there occurs a further curve $\psi_\lambda = 0$, say closed or $\subset$-shaped, lying in the region bounded by $y = 0$, a, by $\lambda = \lambda_{-1}$, λ_0 and by the curve $\psi_\lambda = 0$ shown, and such that its image in the (y, ψ)-plane intersects the part of $\lambda = \lambda_{-1}$ shown broken, then Benjamin's result [(4.13) of his paper] is meaningless, for the integrand is many-valued.

$\{O, V(y), W(y)\}$ and with pressure p_0 on the axis, we denote its stream function by $\psi_A(y)$, its total-head pressure by $\rho H_A(y)$, and its circulation by $\{8\pi^2 I_A(y)\}^{\frac{1}{2}}$. A total-head function $H(\psi)$ and a circulation function $I(\psi)$ are then defined parametrically by

$$\left.\begin{aligned} \psi = \psi_A(\tau) &\equiv \int_0^\tau W(t)\,dt, \\ H = H_A(\tau) &\equiv \left\{\frac{p_0}{\rho} + \int_0^\tau \frac{V^2(t)}{2t}\,dt\right\} + \tfrac{1}{2}\{V^2(\tau) + W^2(\tau)\}, \\ I = I_A(\tau) &\equiv \tau V^2(\tau), \\ 0 \leqslant \tau \leqslant a, \quad 0 \leqslant \psi &\leqslant b, \quad b \equiv \int_0^a W(t)\,dt, \end{aligned}\right\} \tag{1.1}$$

where τ is a parameter, and where it is assumed that $W > 0$ on the closed interval $[0, a]$. Then (see, for example, Squire 1956 or Benjamin 1962) possible steady flows in the pipe are governed by

$$\Psi_{yy} + \tfrac{1}{2}y^{-1}\Psi_{zz} = H'(\Psi) - \tfrac{1}{2}y^{-1}I'(\Psi), \tag{1.2a}$$

with

$$\Psi(0, z) = 0, \quad \Psi(a, z) = b, \tag{1.2b}$$

dashes denoting derivatives of functions of one variable.

Obviously $\psi_A(y)$ satisfies (1.2); Benjamin points out that other functions $\psi(y)$ may also satisfy (1.2), and thus represent cylindrical flows which are said to be *conjugate* to A. He gives examples of such conjugate flows; moreover, it is well known that non-linear boundary-value problems like (1.2), with $\Psi_{zz} \equiv 0$, often have more than one solution.

We suppose throughout this paper that the given primary flow A *cannot* support standing waves of small amplitude; in other words, we assume that if we substitute

$$\Psi(y, z) = \psi_A(y) + \epsilon\, e^{i\alpha z} s(y)$$

into (1.2), to obtain upon linearization with respect to ϵ

$$s_{yy} - \{\tfrac{1}{2}y^{-1}\alpha^2 + H''(\psi_A) - \tfrac{1}{2}y^{-1}I''(\psi_A)\}s = 0, \tag{1.3a}$$

with

$$s(0) = 0, \quad s(a) = 0, \tag{1.3b}$$

then the linear eigenvalue problem (1.3) has no non-negative eigenvalues α^2. Let B denote the conjugate flow 'nearest' A in a sense which will presently be made precise. The following two ideas are crucial to Benjamin's theory of vortex breakdown.

(i) The conjugate flow B *can* support infinitesimal standing waves; that is, if in (1.3) we replace ψ_A by ψ_B, at least one eigenvalue $\alpha^2 \geqslant 0$ exists.

(ii) The flow force

$$\begin{aligned} S &= 2\pi \int_0^a \left(\frac{p}{\rho} + w^2\right) dy, \\ &= 2\pi \int_0^a \{H(\psi) - \tfrac{1}{2}y^{-1}I(\psi) + \tfrac{1}{2}\psi_y^2\}\,dy, \end{aligned} \tag{1.4}$$

is greater for B than for A.

Benjamin argues that, if $S_B - S_A$ is small, the addition of waves to the cylindrical flow B can reduce its flow force to a value near that of A, and that this makes possible a transition, resembling an undular hydraulic jump, from flow A to a combination of flow B and standing waves. On the other hand, if $S_B - S_A$ is not small, a strong transition, resembling a dissipative hydraulic jump, can reduce H sufficiently to change flow A into flow B, with or without waves.

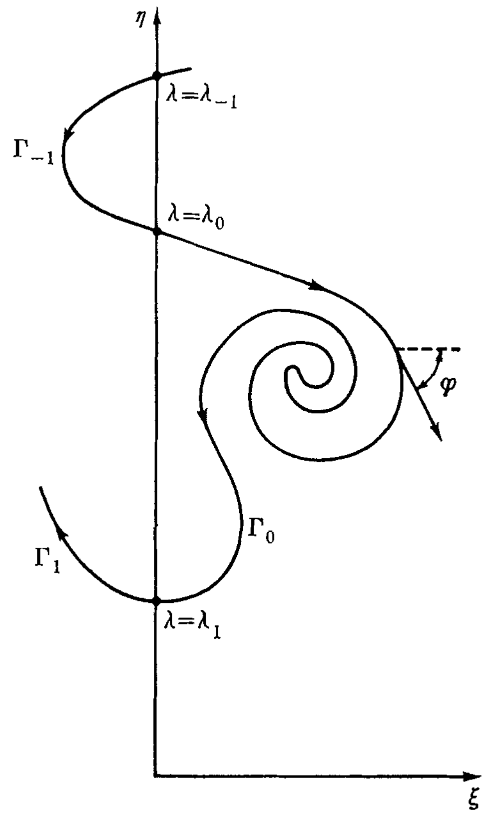

FIGURE 1. Notation for the curve Γ. Arrows along the arcs Γ_n point in the direction of λ increasing.

The present paper is concerned only with results like (i) and (ii), and not with their interpretation. To compare the stream functions $\psi_A(y)$, $\psi_B(y)$, ... of the primary and conjugate flows, we embed them in a continuous one-parameter family consisting of all solutions $\psi(y, \lambda)$ of (1.2a) which are independent of z and vanish on the axis $y = 0$; here λ is a parameter which identifies different solutions and represents the velocity on the axis:

$$\lambda = \psi_y(0, \lambda).$$

The primary and conjugate flows are characterized by particular values λ_n of λ. It turns out that both the results above depend on the nature of a certain curve Γ (figure 1) which is defined parametrically in (say) a (ξ, η)-plane by

$$\xi = \psi(a, \lambda) - b, \quad \eta = \psi_y(a, \lambda).$$

The zeros λ_n of $\xi(\lambda)$ are the values of λ for ψ_A, ψ_B, ... The curve Γ is simple (i.e. it cannot cross itself) and smooth. Let the unit tangent vector to Γ, in the direction of λ increasing, have components $(\cos\varphi, -\sin\varphi)$; we can define $\varphi(\lambda)$ uniquely by specifying that $|\varphi(\lambda_0)| < \frac{1}{2}\pi$ for the primary flow $\psi_A \equiv \psi(y, \lambda_0)$, and by requiring φ to be continuous. It will be shown that angles $\varphi \leqslant -\frac{1}{2}\pi$ are impossible; that conjugate flows $\psi(y, \lambda_n)$ cannot support standing waves if $\varphi(\lambda_n) < \frac{1}{2}\pi$; and that they can support such waves if $\varphi(\lambda_n) \geqslant \frac{1}{2}\pi$. Now, if λ_1 is the zero of $\xi(\lambda)$ nearest λ_0, and if $\lambda_1 > \lambda_0$, then in figure 1 we must have (since $\varphi \leqslant -\frac{1}{2}\pi$ is impossible)

$$\eta(\lambda_1) < \eta(\lambda_0) \quad \text{and} \quad \varphi(\lambda_1) \geqslant \tfrac{1}{2}\pi.$$

Hence flow B, with stream function $\psi(y, \lambda_1)$, can support waves.

Turning to the flow force, we have by (1.4)

$$\begin{aligned} S'(\lambda) &= 2\pi \int_0^a [\{H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi)\}\psi_\lambda + \psi_y \psi_{y\lambda}]\, dy, \\ &= 2\pi[\psi_y \psi_\lambda]_0^a + 2\pi \int_0^a \{H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi) - \psi_{yy}\}\psi_\lambda\, dy, \end{aligned}$$

and so by (1.2)

$$\begin{aligned} S(\lambda_{n+1}) - S(\lambda_n) &= 2\pi \int_{\lambda_n}^{\lambda_{n+1}} \psi_y(a, \lambda)\,\psi_\lambda(a, \lambda)\, d\lambda, \\ &= 2\pi \int_{\Gamma_n} \eta\, d\xi, \end{aligned} \tag{1.5}$$

where Γ_n is the arc $\lambda_n \leqslant \lambda < \lambda_{n+1}$ of Γ. Hence, if λ_1 characterizes flow B, the difference $S_B - S_A$ in flow force is 2π times the area between Γ_0 and the η-axis; this is positive because $\eta(\lambda_1) < \eta(\lambda_0)$.

2. The function $\psi(y, \lambda)$ and its partial derivatives

We consider the function $\psi(y, \lambda)$ defined by the initial-value problem†

$$\psi_{yy} = H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi) \equiv f(y, \psi) \quad \text{(say)}, \tag{2.1a}$$

with

$$\psi(0) = 0, \quad \psi_y(0) = \lambda. \tag{2.1b}$$

The existence and uniqueness of this function for $0 \leqslant y \leqslant a$ and for all bounded real values of λ are not obvious *a priori* because $f(y, \psi)$ has been defined only for $0 \leqslant \psi \leqslant b$, and because f is singular on $y = 0$. We assume‡ that in (1.1) the given functions W and V^2 have continuous first and second derivatives on $[0, a]$; that $W > 0$ there; and that $V^2(y)/2y$ (which is the square of the angular velocity) has a continuous first derivative at $y = 0$. Then it is easy suitably to extend the definition of f to all values of ψ, and to prove the existence of a unique $\psi(y, \lambda)$ on any rectangle

$$R\colon\ 0 \leqslant y \leqslant c, \quad -\Lambda \leqslant \lambda \leqslant \Lambda,$$

where c and Λ are arbitrary positive finite numbers. This is done in the appendix.

† The dependence of ψ on λ will not be displayed where λ is held fixed throughout a calculation.

‡ Derivatives at the left and right end-points of closed intervals are, of course, right-hand and left-hand derivatives, respectively. Any function of form $F(y)/y$ is to be defined at $y = 0$ by its limiting value.

Now in general $\psi(a, \lambda)$ does not vary monotonically with λ (figure 2). Hence in many cases there exists a finite or infinite sequence

$$\{\lambda_n\} \equiv \ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots \quad (\lambda_n < \lambda_{n+1}),$$

such that

$$\psi(a, \lambda_n) = b, \tag{2.2}$$

and the functions $\psi(y, \lambda_n)$ then represent the primary flow A and its conjugates. We choose

$$\lambda_0 = W(0), \quad \text{so that} \quad \psi(y, \lambda_0) \equiv \psi_A(y).$$

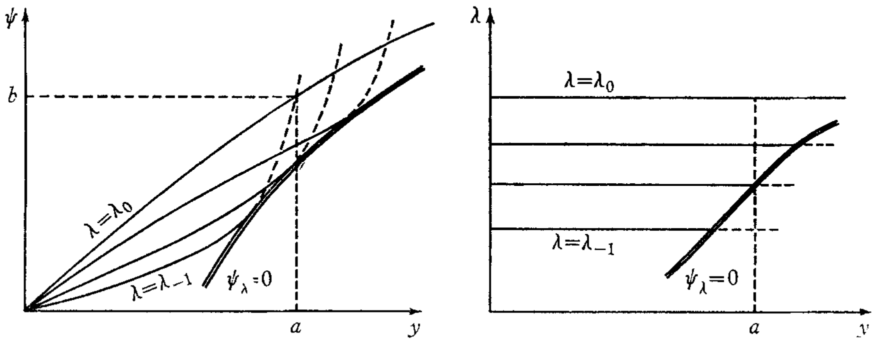

FIGURE 2. A mapping of the (y, λ)-plane on to the (y, ψ)-plane by the function $\psi(y, \lambda)$. Curves $\psi_\lambda = 0$ correspond to folds bounding the various sheets of the (y, ψ)-plane.

Of course, only values $\lambda_n \leqslant \lambda_0$ (or only values $\lambda_n \geqslant \lambda_0$) may exist for a particular function f. Moreover, only those conjugate flows satisfying

$$0 \leqslant \psi(y, \lambda_n) \leqslant b \quad \text{for} \quad 0 \leqslant y \leqslant a, \tag{2.3}$$

are physically significant, since the others depend on the arbitrary extension of $f(y, \psi)$. However, while we may be able to see in particular cases whether (2.3) is satisfied for a given λ_n, there seems to be no way of establishing this in the general theory. (Of course, (2.3) cannot be satisfied if $\lambda_n < 0$.)

If the λ_n have an accumulation point (limit point), there are complications: these could be handled, but the effort does not seem worth while. If $\{\lambda_n\} = \lambda_0$ alone, there is nothing to discuss. We consider only primary flows for which the λ_n are isolated and at least one λ_n differs from λ_0.

Since we are interested in the variation with λ of wave-carrying capacity and of flow force, the derivative $\psi_\lambda \equiv \chi(y, \lambda)$ is of importance. Formally differentiating (2.1), we obtain

$$\chi_{yy} - f_\psi(y, \psi(y, \lambda))\, \chi = 0 \tag{2.4a}$$

with

$$\chi(0) = 0, \quad \chi_y(0) = 1, \tag{2.4b}$$

and it can be shown (see the appendix) that on the rectangle R the solution of the problem (2.4) exists, is unique, is continuous with respect to λ and twice continuously differentiable with respect to y, and is indeed equal to ψ_λ.

We now assign to each value of λ an integer which will be shown to describe the standing waves which the flow associated with $\psi(y, \lambda)$ can support. (It is helpful to regard each function $\psi(y, \lambda)$, with λ fixed, as representing a flow, even though for $\lambda \neq \lambda_n$ the flux $\psi(a, \lambda)$ differs from that of the primary flow.)

Let the λ-derivative $\psi_\lambda(y,\lambda)$ have m zeros ($m = 0, 1, 2, \ldots$) on the half-open y-interval $(0, a]$, for a given value of λ; then $\psi(y,\lambda)$ is said to be of type m, and we write $T(\lambda) = m$.

Figure 3 illustrates this definition. To grasp its significance, consider the following initial-value problem, which is closely related to (1.3) and (2.4).

$$s_{yy} - \{\tfrac{1}{2}y^{-1}\alpha^2 + f_\psi(y, \psi(y,\lambda))\}\, s = 0 \tag{2.5a}$$

with

$$s(0) = 0, \quad s_y(0) = 1. \tag{2.5b}$$

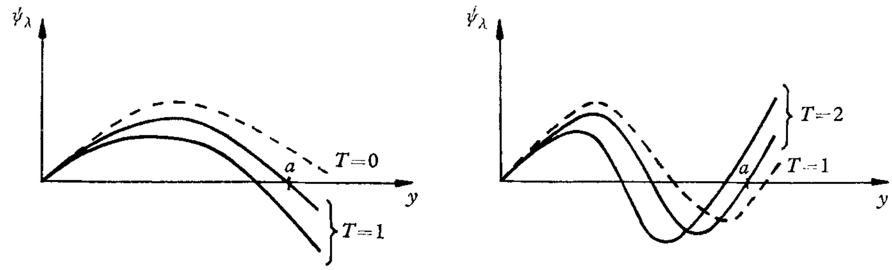

FIGURE 3. Curves of $\psi_\lambda(y, \lambda)$ which illustrate the definition of type. The ordinate and slope at $y = a$ are $\xi'(\lambda)$ and $\eta'(\lambda)$, respectively.

Let $y_r(\alpha^2, \lambda)$, where $r = 1, 2, \ldots$, denote the rth positive zero of $s(y, \alpha^2, \lambda)$, such that $y_r < y_{r+1}$. The existence proof for (2.4) also covers (2.5), since both equations have the same type of singularity at $y = 0$, and it follows from the usual arguments (e.g. Burkill 1956; Coddington & Levinson 1955) that each y_r is a continuous increasing function of α^2 and continuous with respect to λ. The definition of type $m \geqslant 1$ states that

$$y_m(0,\lambda) \leqslant a \quad \text{and} \quad y_{m+1}(0,\lambda) > a. \tag{2.6}$$

By increasing α^2 we increase each y_r; hence there exists an eigenvalue $\alpha^2 \geqslant 0$ such that $y_m(\alpha^2, \lambda) = a$ [and then $s(a) = 0$, as required in (1.3)]. Accordingly, *if ψ is of type 0, no standing waves are possible. If ψ is of type $m \geqslant 1$, the corresponding flow can support standing waves, for the longest of which (associated with the smallest eigenvalue $\alpha^2 \geqslant 0$) the amplitude function $s(y)$ has m zeros on $(0, a]$.*

Our definition of type 0 and type $m \geqslant 1$ corresponds to Benjamin's definition of *supercritical* and *subcritical*, respectively, unless $y_1(0,\lambda) = a$. In that case ψ is of type 1, but Benjamin calls it *critical*.

3. The curve Γ

The curve Γ is defined parametrically by

$$\xi = \psi(a,\lambda) - b, \quad \eta = \psi_y(a,\lambda), \tag{3.1}$$

and may be described as follows.

(i) *The curve is simple (it cannot cross itself).* For if we had

$$\psi(a,\lambda) = \psi(a,\mu) \quad \text{and} \quad \psi_y(a,\lambda) = \psi_y(a,\mu) \quad \text{with} \quad \lambda \neq \mu,$$

then by the ordinary uniqueness theorem of differential equations we would have $\psi(y,\lambda) = \psi(y,\mu)$ on every y-interval $[\epsilon, a]$ ($\epsilon > 0$). But (2.1) shows that

$$\psi(y,\lambda) \neq \psi(y,\mu) \quad \text{for} \quad \lambda \neq \mu$$

and for sufficiently small positive values of y.

(ii) *The curve is smooth: that is, the functions $\xi(\lambda)$ and $\eta(\lambda)$ have continuous derivatives ξ' and η' such that $\xi'^2+\eta'^2 \neq 0$.* The continuity of ψ_λ and $\psi_{\lambda y}$ was noted after (2.4). If we had

$$\psi_\lambda(a,\lambda) = 0 \quad \text{and} \quad \psi_{\lambda y}(a,\lambda) = 0$$

for some particular value of λ, then we would have $\psi_\lambda = 0$ on every y-interval $[\epsilon, a]\,(\epsilon > 0)$. But (2.4) shows that $\psi_\lambda \neq 0$ for sufficiently small positive values of y.

(iii) *If $\psi(y,\lambda_n)$ is of even type, with $\xi'(\lambda_n) \neq 0$, the open arc $\tilde{\Gamma}_n\colon \lambda_n < \lambda < \lambda_{n+1}$ lies in $\xi > 0$ and $\tilde{\Gamma}_{n-1}$ lies in $\xi < 0$.* By the definition of λ_n, and because even type implies that $\psi_\lambda(a,\lambda) \geqslant 0$, we have

$$\xi(\lambda_n) = 0 \quad \text{and} \quad \xi'(\lambda_n) \equiv \psi_\lambda(a,\lambda_n) > 0.$$

Hence $\xi > 0$ for $\lambda - \lambda_n > 0$ and sufficiently small. If we had $\xi(\lambda) = 0$ for some λ on $\tilde{\Gamma}_n$ this would contradict the definition of λ_{n+1}.

(iv) We define the angle $\varphi(\lambda)$ by

$$\cos\varphi = \xi'/(\xi'^2+\eta'^2)^{\frac{1}{2}}, \quad \sin\varphi = -\eta'/(\xi'^2+\eta'^2)^{\frac{1}{2}}, \tag{3.2}$$

by requiring φ to be continuous [which is possible by (ii)], and by specifying (since the primary flow has type 0, so that $\xi'(\lambda_0) > 0$) that $|\varphi(\lambda_0)| < \frac{1}{2}\pi$. Then

$$\left.\begin{aligned} &\textit{$\psi(y,\lambda)$ is of type m, that is $T(\lambda) = m$, if and only if} \\ &\qquad\qquad (m-\tfrac{1}{2})\,\pi \leqslant \varphi(\lambda) < (m+\tfrac{1}{2})\,\pi. \end{aligned}\right\} \tag{3.3}$$

To prove this statement, we first observe that it is true at λ_0. Moreover, if $T(\mu) \neq T(\nu)$, with $\mu < \nu$, then $\psi_\lambda(a,\lambda)$ must vanish somewhere on $[\mu,\nu]$ because of the continuity of the zeros $y_r(\lambda)$ of $\psi_\lambda(y,\lambda)$; that is, $\xi' = 0$ and $\cos\varphi = 0$ somewhere on $[\mu,\nu]$. Consider now the behaviour of $T(\lambda)$ as λ increases or decreases from λ_0 and passes through zeros of $\xi'(\lambda)$. It is clear (see figure 3) that where T is even we have $\xi' \geqslant 0$, and where T is odd we have $\xi' \leqslant 0$; also that where T changes from $2p$ to $2p+1$ or from $2p+1$ to $2p$ (p denoting any integer $\geqslant 0$) we have $\psi_{\lambda y}(a,\lambda) < 0$, that is $\eta'(\lambda) < 0$, and where T changes from $2p+1$ to $2p+2$ or from $2p+2$ to $2p+1$ we have $\eta' > 0$. Hence for isolated zeros of $\xi'(\lambda)$ across which $\xi'(\lambda)$ changes sign, only the four changes of type shown in figure 4 are possible. For these T increases or decreases according as φ increases or decreases through an odd multiple of $\frac{1}{2}\pi$, the larger type being always assigned at the critical value of λ.

This does not prove (3.3) because $\xi'(\lambda)$ may have other kinds of zero.† Accordingly let λ_* denote any zero of $\xi'(\lambda)$ on $[\lambda_0,\infty)$ and assume for definiteness that $T(\lambda_*) = 2p+1$, so that

$$y_{2p+1}(\lambda_*) = a \quad \text{and} \quad \eta'(\lambda_*) \equiv \psi_{\lambda y}(a,\lambda_*) < 0.$$

By the continuity of $\psi_{\lambda y}$, there exists a number $\delta > 0$ such that on the closed interval $I_\delta\colon [\lambda_* - \delta,\, \lambda_* + \delta]$ we have

$$\eta' < 0, \quad y_{2p} < a, \quad y_{2p+2} > a,$$

† For example, if λ_* is an accumulation point of zeros of $\xi'(\lambda)$ (which here implies that λ_* itself is a zero) there may exist sequences $\{\lambda_{(k)}\}$ converging to λ_* from below such that $T(\lambda_{(k)})$ does not tend to any limit as $k \uparrow \infty$ and $\lambda_{(k)} \uparrow \lambda_*$.

and
$$\left.\begin{aligned} T &= 2p \quad \text{wherever} \quad \xi' > 0, \\ T &= 2p+1 \quad \text{wherever} \quad \xi' \leqslant 0. \end{aligned}\right\} \tag{3.4}$$

Assume that the rule (3.3) holds at $\lambda_* - \delta$; since by (3.4) the type there is either $2p$ or $2p+1$, we have
$$(2p-\tfrac{1}{2})\,\pi \leqslant \varphi(\lambda_* - \delta) < (2p+\tfrac{3}{2})\,\pi.$$

Then, since $\eta' < 0\,(\sin\varphi > 0)$ on I_δ, and since φ is continuous,
$$2p\pi < \varphi(\lambda) < (2p+1)\,\pi \quad \text{on} \quad I_\delta.$$

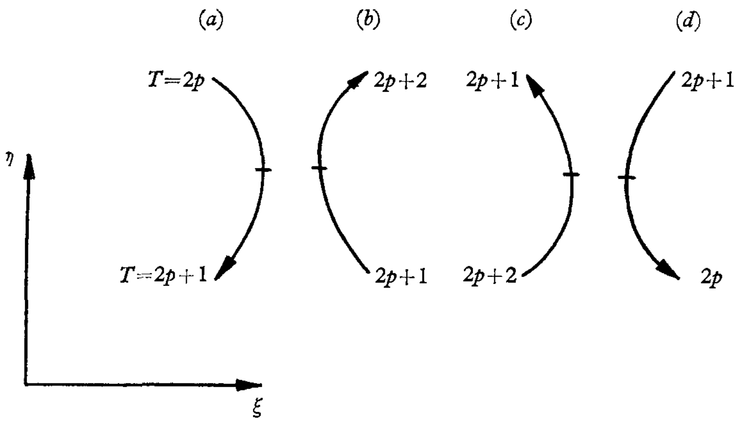

FIGURE 4. Local forms of the curve Γ for changes of the type $T(\lambda)$. Arrows point in the direction of λ increasing.

Hence
$$\left.\begin{aligned} &\xi' > 0\,(\cos\varphi > 0) \quad \text{wherever} \quad 2p\pi < \varphi < (2p+\tfrac{1}{2})\,\pi, \\ &\xi' \leqslant 0 \quad \text{wherever} \quad (2p+\tfrac{1}{2})\,\pi \leqslant \varphi < (2p+1)\,\pi, \end{aligned}\right\} \text{ on } I_\delta. \tag{3.5}$$

From (3.4) and (3.5) we observe that, if the rule (3.3) holds on $[\lambda_0, \lambda_* - \delta]$, it holds on $[\lambda_0, \lambda_* + \delta]$. Since it holds in a neighbourhood of λ_0, it may be extended in this way to any bounded value of λ.

(v) *Only angles $\varphi(\lambda) > -\frac{1}{2}\pi$ are possible.* For, if $\varphi \downarrow -\frac{1}{2}\pi$, then in the limit
$$\psi_\lambda(a,\lambda) = 0, \quad \psi_\lambda > 0 \quad \text{for} \quad 0 < y < a, \quad \text{and} \quad \psi_{\lambda y}(a,\lambda) > 0.$$

This is impossible.

We can now state our principal results. If λ_{-1} exists, consider $\tilde{\Gamma}_{-1}$: $\lambda_{-1} < \lambda < \lambda_0$, which lies in $\xi < 0$ by (iii); if λ_1 exists, consider $\tilde{\Gamma}_0$: $\lambda_0 < \lambda < \lambda_1$, which lies in $\xi > 0$ by (iii). Since $\varphi > -\frac{1}{2}\pi$, it is intuitively obvious from figure 1 that
$$\left.\begin{aligned} &\text{for} \quad j = \pm 1, \quad \tfrac{1}{2}\pi \leqslant \varphi(\lambda_j) \leqslant \tfrac{3}{2}\pi, \\ &\text{with} \quad \eta(\lambda_{-1}) > \eta(\lambda_0) \quad \text{and/or} \quad \eta(\lambda_0) > \eta(\lambda_1), \end{aligned}\right\} \tag{3.6}$$

and this can be proved (see the appendix). Hence *for $j = \pm 1$, the conjugate stream function $\psi(y, \lambda_j)$ is of type 1 unless $\varphi(\lambda_j) = \frac{3}{2}\pi$, when it is of type 2; in either case the corresponding flow can support waves. Moreover, by* (1.5) *we have $S(\lambda_j) > S(\lambda_0)$.* Note that in these respects the (possibly tortuous) arc Γ_0 is equivalent to the uncomplicated one of figure 4 (*a*), while Γ_{-1} corresponds to the arc of figure 4 (*d*).

If $\psi(y,\lambda_n)$ is of type $m \geqslant 1$ and if $\xi'(\lambda_n) \neq 0$ there are four possibilities for Γ_n, according to whether m is even or odd and whether $\eta(\lambda_{n+1}) \gtrless \eta(\lambda_n)$. They correspond to the four cases in figure 4, and in each case the flow force increases when the type increases.

A comment on the possibility $\xi'(\lambda_n) = 0$, which represents a type of local linearity, may be appropriate. If $f(y,\psi)$ is linear in ψ, so that (2.1*a*) becomes

$$\psi_{yy} - \{H'' - \tfrac{1}{2}y^{-1}I''\}\psi = H'(0), \quad H'' = \text{const.}, \; I'' = \text{const.},$$

and if the corresponding problem (1.3) has an eigenvalue $\alpha^2 = 0$, then conjugate solutions result from the addition to ψ_A of eigensolutions of arbitrary amplitude, and the curve Γ degenerates to the η-axis. This situation has been excluded from the present paper (both by our assumption that ψ_A has type 0, and by our assumption that the λ_n are isolated), but it is simulated locally, for small values of $\lambda - \lambda_n$, if $\xi'(\lambda_n) = 0$. Then solutions

$$\psi(y,\lambda_n+\epsilon) = \psi(y,\lambda_n) + \epsilon\psi_\lambda(y,\lambda_n) + o(\epsilon),$$

in which ψ_λ satisfies a linear equation, are conjugate within an error of $o(\epsilon)$ for arbitrary (small) values of ϵ.

4. A further property of the conjugate stream functions $\psi(y,\lambda_1)$ and $\psi(y,\lambda_{-1})$

Assume that λ_1 exists and let $\psi(y,\lambda_1) \equiv \psi_I(y)$. In this section we make a comparison with $\psi(y,\lambda_0) \equiv \psi_A(y)$ which shows that

$$\psi_I(y) > \psi_A(y) \quad (0 < y < a), \tag{4.1a}$$

$$\psi_{Iy}(0) > \psi_{Ay}(0), \quad \psi_{Iy}(a) < \psi_{Ay}(a). \tag{4.1b, c}$$

Similar inequalities, but in the opposite direction, hold for $\psi(y,\lambda_{-1})$ relative to ψ_A.

Benjamin (1962) used the property that $\psi_A(y)$ and $\psi_B(y)$ intersect only at $y = 0, a$ as one of several definitions of the conjugate flow B 'adjacent' to A; equivalence of these definitions was not proved. Our definition of $\psi(y,\lambda_1)$ implies (4.1*a*), but it is not obvious whether the inequality $\psi(y,\lambda_n) > \psi(y,\lambda_0)$ on $(0,a)$ implies $\lambda_n = \lambda_1$ uniquely in all cases.

Let $\psi(y,\lambda) - \psi(y,\lambda_0) = \phi(y,\lambda)$. Because $\psi(y,\lambda_0)$ is of type 0, we have $\phi_\lambda(y,\lambda_0) > 0$ on $(0,a]$; also $\phi_{y\lambda}(0,\lambda) = 1$. Hence there exists a number $\lambda_* > \lambda_0$ such that

$$\phi(y,\lambda_*) > 0 \quad (0 < y \leqslant a). \tag{4.2}$$

Also, by (2.1*b*) and § 3 (iii),

$$\phi(0,\lambda) = 0, \quad \phi_y(0,\lambda) = \lambda - \lambda_0, \tag{4.3}$$

and

$$\phi(a,\lambda) > 0, \quad \lambda_* \leqslant \lambda < \lambda_1. \tag{4.4}$$

The function $\phi(y,\lambda)$ is continuously differentiable on the rectangle R, and hence also on

$$R_*: 0 \leqslant y \leqslant a, \quad \lambda_0 < \lambda_* \leqslant \lambda \leqslant \lambda_1.$$

Assume that (4.1 *a*) is false, so that there exists at least one point $\tilde{y}$ on $(0,a)$ such that

$$\phi(\tilde{y},\lambda_1) \leqslant 0. \tag{4.5}$$

We now prove a result made plausible by sketching the graph of ϕ versus y for various values of λ; namely, that (4.2) to (4.5) imply the existence on R_* of at least one point $(\hat{y}, \hat{\lambda})$ at which $\phi = \phi_y = 0$. But this is impossible by the uniqueness argument used in § 3 (i); hence (4.5) is false, and (4.1 a) is true.

By (4.3) there exists a number $y_0 > 0$ such that on R_*

$$\phi \geqslant \tfrac{1}{2}(\lambda_* - \lambda_0)\, y \quad \text{for} \quad y \leqslant y_0.$$

Define on $[\lambda_*, \lambda_1]$,

$$m(\lambda) = \min\{\phi(y,\lambda);\; y_0 \leqslant y \leqslant a\}.$$

Then $m(\lambda)$ is continuous, with $m(\lambda_*) > 0$ and $m(\lambda_1) \leqslant 0$; hence there exists $\hat{\lambda}$ such that $m(\hat{\lambda}) = 0$, and $\hat{y} > y_0$ such that

$$\phi(\hat{y}, \hat{\lambda}) = m(\hat{\lambda}) = 0.$$

It remains to show that $\hat{y} < a$, for then the minimum occurs at an interior point of $[y_0, a]$ and $\phi_y = 0$ there. If $m(\lambda_1) < 0$, then $\hat{\lambda} < \lambda_1$ and $\hat{y} < a$ by (4.4). If $m(\lambda_1) = 0$, then equality must hold in (4.5), and we can take $\hat{y} = \tilde{y} < a$.

The result (4.1 b) follows from (4.3). As for (4.1 c): equality there would make $\phi = \phi_y = 0$ at (a, λ_1), and the opposite inequality would imply (4.5).

Appendix. Various mathematical details

I. *The function* $f(y, \psi)$

Our first task is to extend the definitions (1.1) of the functions $H(\psi)$ and $I(\psi)$. To this end, we first extend the definitions of $W(\tau)$ and $V^2(\tau)$ to all (real) values of τ in any manner which (a) makes W, V^2, τV^2 and their first and second derivatives bounded and uniformly continuous for all τ; (b) makes $W \geqslant \delta$ for some $\delta > 0$ which is independent of τ; and (c) leaves V^2/τ continuously differentiable at $\tau = 0$. (We have already assumed that such conditions hold on $[0, a]$.) For example, we could write, for $\tau \leqslant 0$,

$$W(\tau) = W(0) + e^{-\tau^2}\{W'(0)\,\tau + \tfrac{1}{2}W''(0)\,\tau^2 - C\tau^3\},$$

and choose the constant C sufficiently large to make $W(\tau) \geqslant W(0)/2$ for $\tau \leqslant 0$. Then $H_A(\tau)$ and $I_A(\tau)$ are defined and suitably differentiable for all values of τ, and the relation $\psi = \psi_A(\tau)$ has a unique inverse $\tau = \tau(\psi)$ for all ψ. Since

$$H(\psi) = H_A(\tau), \quad I(\psi) = I_A(\tau), \quad d/d\psi = \{1/W(\tau)\}\,(d/d\tau),$$

the functions $H(\psi)$, $I(\psi)$ and their first and second derivatives are bounded and uniformly continuous for all ψ.

Consider the function

$$f(y, \psi) = H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi) \quad \text{on} \quad D\colon 0 < y \leqslant c, \quad -\infty < \psi < \infty,$$

where ψ is regarded as independent of y at this stage, and c is an arbitrary positive finite number. We define

$$K = \max\{c|H''(\psi)| + \tfrac{1}{2}|I''(\psi)|\},$$

and note that there exists a function $Z(\alpha)$, independent of ψ and tending to zero with α, such that

$$c|H''(\psi+\alpha) - H''(\psi)| + \tfrac{1}{2}|I''(\psi+\alpha) - I''(\psi)| \leqslant Z(\alpha).$$

This yields two fundamental inequalities:

$$|f_\psi(y,\psi)| \leqslant K/y \quad \text{on} \quad D, \tag{A 1}$$

$$|f_\psi(y,\psi+\alpha)-f_\psi(y,\psi)| \leqslant Z(\alpha)/y \quad \text{on} \quad D. \tag{A 2}$$

To obtain a third, we write

$$\psi = y\theta, \quad f(y,y\theta) = g(y,\theta),$$

so that
$$g(y,\theta) = H'(0)+\int_0^\theta \{yH''(y\sigma)-\tfrac{1}{2}I''(y\sigma)\}\,d\sigma.$$

With a little manipulation, it follows that

$$|g(y+\beta,\theta+\gamma)-g(y,\theta)| \leqslant |\theta|\{c^{-1}K|\beta|+Z(\beta\theta)\}+K|\gamma| \quad \text{on} \quad D. \tag{A 3}$$

II. *The initial-value problem for $\psi(y,\lambda)$*

If we write

$$\psi = \psi_A(y)+\phi, \quad f(y,\psi_A(y)+\phi)-f(y,\psi_A(y)) = F(y,\phi),$$

so that

$$F(y,0) = 0, \quad |F(y,\phi_*)-F(y,\phi)| \leqslant K|\phi_*-\phi|/y, \tag{A 4}$$

the initial-value problem (2.1) for $\psi(y,\lambda)$ becomes

$$\phi_{yy} = F(y,\phi),$$

with
$$\phi(0) = 0, \quad \phi_y(0) = \lambda - W(0) \equiv \mu \quad \text{(say)}.$$

We apply the usual Picard scheme of successive approximation (Burkill 1956; Coddington & Levinson 1955):

$$\phi_0(y) = \mu y,$$

$$\phi_{n+1}(y) = \mu y+\int_0^y ds\int_0^s F(t,\phi_n(t))\,dt, \quad (n \geqslant 0). \tag{A 5}$$

Defining
$$B(x) = 1+\frac{x}{2!}+\ldots+\frac{x^n}{n!(n+1)!}+\ldots = \frac{I_1(2\sqrt{x})}{\sqrt{x}},$$

where I_1 is the modified Bessel function of the first kind of order one, we readily find by induction that on $[0,c]$

$$|\phi_{n+1}(y)-\phi_n(y)| \leqslant |\mu|\frac{K^{n+1}y^{n+2}}{(n+1)!\,(n+2)!}, \quad |\phi_n(y)| \leqslant |\mu|\,yB(Ky). \tag{A 6}$$

The identity implied by (A 5)

$$\theta_n(y) \equiv \frac{\psi_n(y)}{y} = \int_0^1 W(y\sigma)\,d\sigma+\mu+\int_0^1 d\sigma\int_0^{y\sigma} F(t,\phi_{n-1}(t))\,dt$$

shows, in view of the bounds (A 4) and (A 6), that θ_n is continuous on $[0,c]$, and it then follows from (A 3) that $F(y,\phi_n(y))$ is also continuous there.

Therefore we have the usual situation: the sequences $\{\phi_n\}$, $\{F(t,\phi_n(t))\}$ of continuous functions converge uniformly on $[0,c]$, and all the details of the existence and uniqueness proofs for non-singular equations follow here, without restriction on the range of ψ.

III. *The existence and continuity of ψ_λ*

Consider the problem

$$\chi_{yy}-y^{-1}h(y)\,\chi = 0, \quad \text{where} \quad h = yH''(\psi(y,\lambda))-\tfrac{1}{2}I''(\psi(y,\lambda)), \tag{A 7a}$$

with
$$\chi(0) = 0, \quad \chi_y(0) = 1, \tag{A 7b}$$

without identifying χ with ψ_λ in the first instance. It is clear that, for fixed λ, h is continuous on $[0, c]$; in fact by (A 5) we have, using the bounds (A 4) and (A 6),

$$k \equiv \max|\psi_y| \leqslant \max|W| + |\mu| + \int_0^c \frac{K}{t}\,|\mu|\,tB(Kt)\,dt,$$

and the definitions of K and $Z(\alpha)$ then show that

$$|h(y)| \leqslant K,$$
$$|h(y+\beta)-h(y)| \leqslant c^{-1}K|\beta| + Z(k\beta).$$

The existence and uniqueness proof for (A 7) by the Picard scheme now go through as before, χ_n/y being a continuous function of y. Moreover, it is straightforward to adapt the relevant theorem of the ordinary theory (Coddington & Levinson 1955, p. 25) to show that ψ_λ equals χ and is continuous with respect to λ.

IV. *Proof of the result* (3.6)

Let $$C\colon \xi = \xi(t), \quad \eta = \eta(t), \quad \alpha \leqslant t \leqslant \beta,$$

be the parametric representation of a closed, simple, piecewise-smooth curve; let C be smooth at $t = \alpha$ (and hence at $t = \beta$, which maps on to the same ξ, η). We may define $\varphi(t)$ by (3.2), by requiring its jumps at the corners of C to have magnitude $\leqslant \pi$, and by requiring it to be continuous elsewhere. Then

$$\varphi(\beta) - \varphi(\alpha) = \pm 2\pi$$

according as t increasing corresponds to the clockwise or anticlockwise direction around C. [For, given $\epsilon > 0$, we can construct a polygon whose points and whose φ depart from those of C by less than ϵ; and, for a polygon with interior angles γ_k at its n vertices, the increase in φ at a vertex is $\pi - \gamma_k$ for the clockwise direction, while $\Sigma\gamma_k = (n-2)\pi$.]

Now form the union of Γ_0 and the segment (say E) of the η-axis between $\eta(\lambda_0)$ and $\eta(\lambda_1)$; this union is a closed, simple, piecewise-smooth curve. Accordingly, identifying $t = \alpha$ with $\lambda = \lambda_0+$, we can relate $\varphi(\lambda_1-)$ to the final angle $\varphi(\lambda_0+) \pm 2\pi$ by way of the straight path E, instead of relating it to the initial angle $\varphi(\lambda_0+)$ by way of the possibly tortuous path Γ_0. Any possibility other than (3.6) then leads to a contradiction.

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